A Berry–Esseen Theorem for Quantum Lattice Systems and the Equivalence of Statistical Mechanical Ensembles

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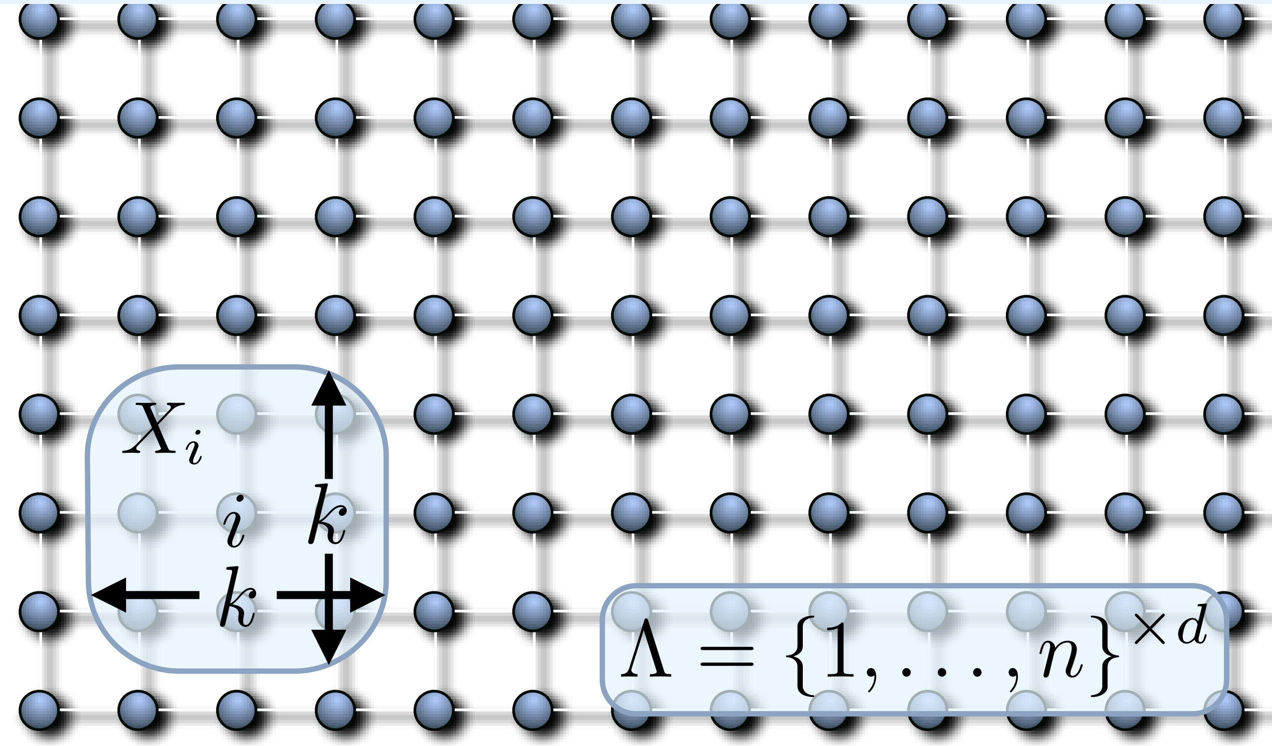
$$X = \sum_{i=1}^{N} X_i$$

Berry-Esseen: $\sup_x |F(x) - G(x)| \leq \frac{C}{\sqrt{N}}$

Central limit theorem:

$$\mathbb{P}[X \leq x] = F(x) \xrightarrow[N\to\infty]{} G(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} \mathrm{d}y \, \mathrm{e}^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

$\mu = \langle X \rangle, \;\; \sigma^2 = \langle (X-\mu)^2 \rangle$



$$X = \sum_{i \in \Lambda} X_i = \sum_k x_k |k\rangle\langle k|$$

X_i bounded and k-local
ϱ sufficiently clustering

Central limit theorem (quantum):

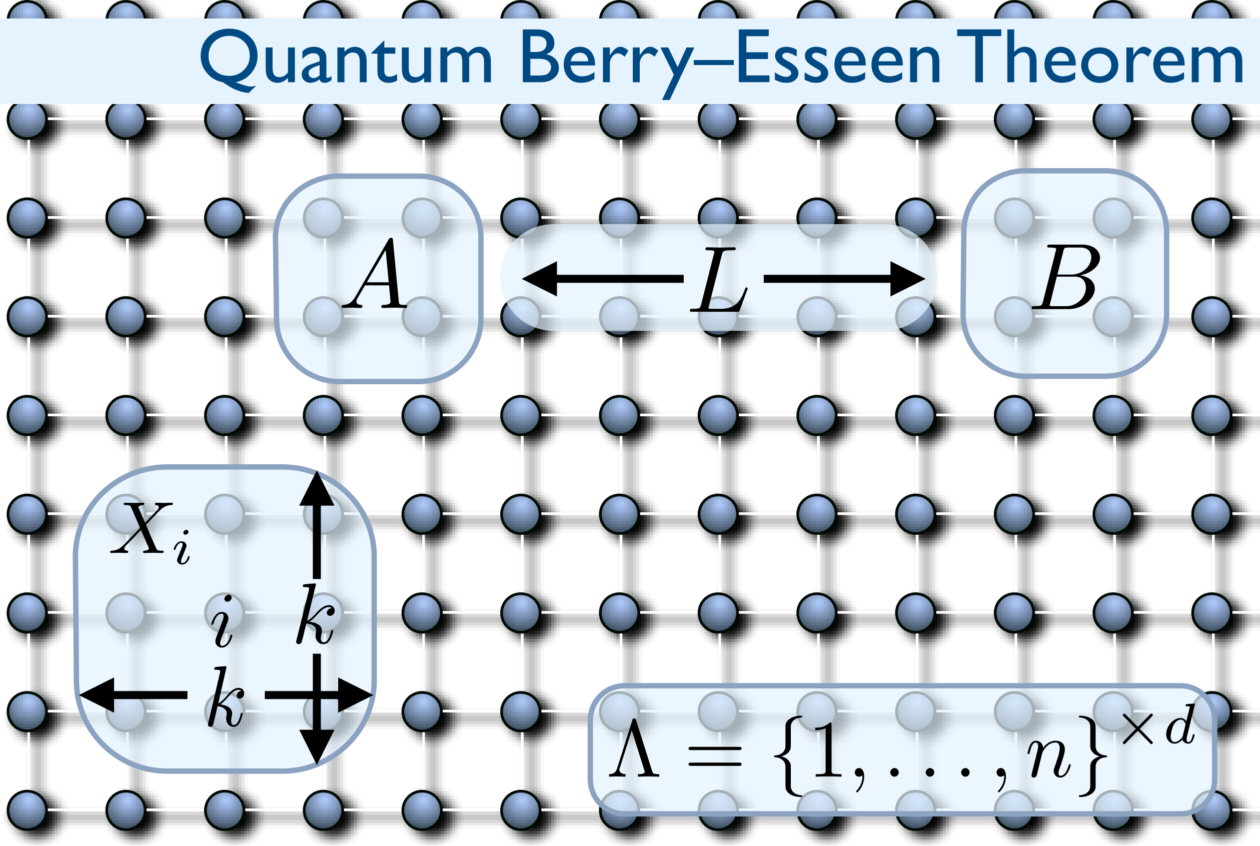
$$\sum_{x_k \leq x} \langle k|\varrho|k\rangle = F(x) \xrightarrow[N \to \infty]{} G(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} \mathrm{d}y \, \mathrm{e}^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

Goderis, Vets (1989); Hartmann, Mahler, Hess (2004)

relation to density of states for $X = H$, $\varrho = \frac{\mathbb{I}}{2^N}$:

$$F(E) - F(E - \Delta E) \propto |\{k : E - \Delta E < E_k \leq E\}|$$

$\mu = \langle X \rangle, \;\; \sigma^2 = \langle (X - \mu)^2 \rangle$



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$$\frac{|\langle AB\rangle - \langle A\rangle\langle B\rangle|}{\|A\|\|B\|} \leq N^z \mathrm{e}^{-L/\xi}$$

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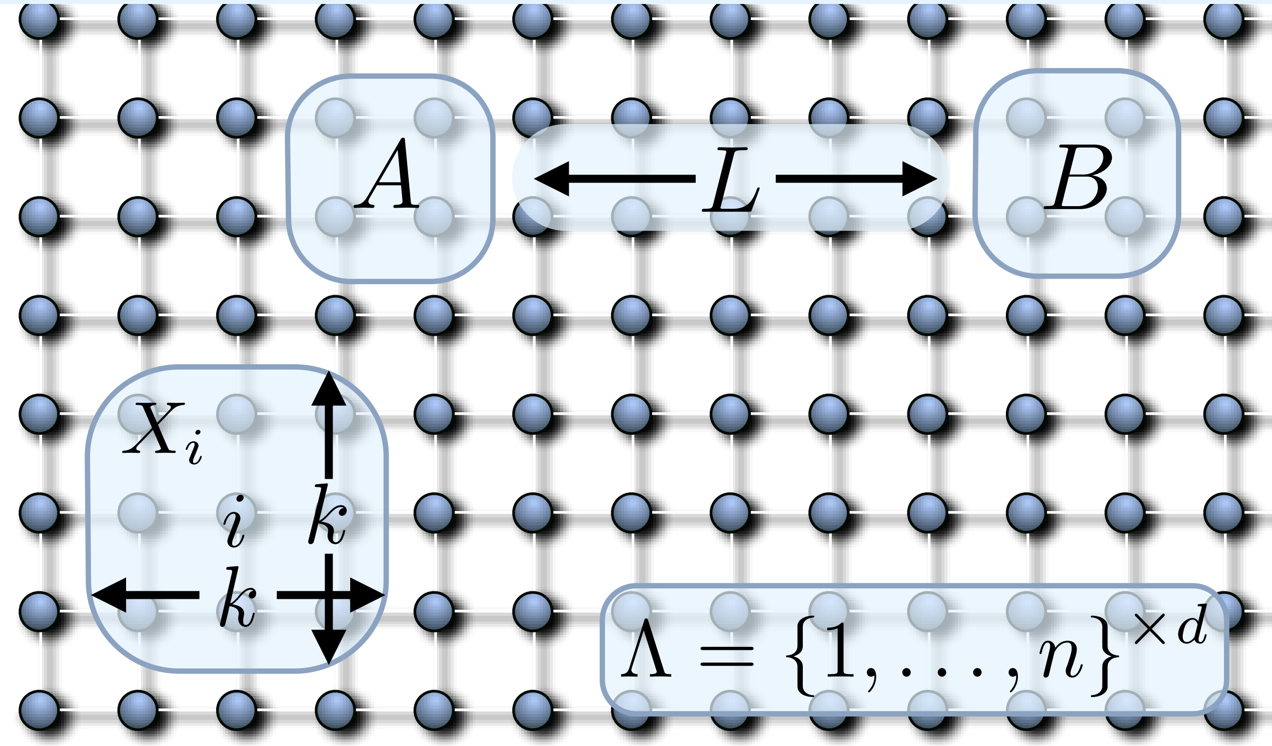

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$$\sup_x |F(x) - G(x)| \le C \frac{\ln^{2d}(N)}{\sqrt{N}}$$

$$C = C_d \frac{(\max\{k,\xi\}(z+1))^{2d}}{\sigma/\sqrt{N}} \max\left\{\frac{1}{\max\{k,\xi\}(z+1)\ln(N)}, \frac{1}{\sigma^2/N}\right\}$$

$$\mu = \langle X\rangle, \quad \sigma^2 = \langle (X-\mu)^2\rangle$$

main ingredient
(also for (quantum) central limit):

$$\sup_x |F(x) - G(x)| \leq \frac{c_1}{T} + \int_0^T \mathrm{d}t \, \frac{|\phi(t) - \mathrm{e}^{-\sigma^2 t^2/2 + \mathrm{i}\mu}|}{|t|}$$

Esseen (1945)

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Esseen (1945)

$\longrightarrow$ bound $|\phi(t) - \mathrm{e}^{-\sigma^2 t^2/2 + \mathrm{i}\mu}|$

$\phi(t) = \langle \mathrm{e}^{\mathrm{i}Xt} \rangle$

- characteristic function
- $X = H$:
  - pure state: Loschmidt echo
  - $\varrho = \frac{\mathbb{I}}{2^N}$: Fourier transform of d.o.s

$\longrightarrow$ bound $\left|\phi(t) - \mathrm{e}^{-\sigma^2 t^2/2 + \mathrm{i}\mu}\right|$

set up differential equation for ϕ and bound its derivative

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set up differential equation for ϕ and bound its derivative

Let $h \in T$ and $k \in T \setminus \{1\}$ be such that $2kh + 1 < n$. For $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, k$, $r = 2, 3, \ldots, k$, and $l = 1, 2, \ldots, r-1$, we let

$$A_j = X_j/B_n,\ Z_j^{(i)} = \sum_{|p-j| \leqslant ih} A_p,\ Z_j^{(0)} = A_j,\ z_j^{(i)} = Z_n - Z_j^{(i)},$$

$$z_j^{(0)} = Z_n,\ \xi_j^{(l)} = e^{it(z_j^{(l-1)} - z_j^{(l)})} - 1,\ a_j^{(r-1)} = \mathbf{E}\left(A_j \prod_{l=1}^{r-1} \xi_j^{(l)}\right),$$

$$\eta_j^{(r)} = e^{-itZ_j^{(r)}} - 1,\quad f_n(t) = \mathbf{E}\, e^{itZ_n}.$$

Since for $j = 1, 2, \ldots, n$ and $r = 2, 3, \ldots, k$

$$\mathbf{E}\, e^{itz_j^{(r)}} = \mathbf{E}(\eta_j^{(r)} + 1) f_n(t) + \mathbf{E}[(\eta_j^{(r)} - \mathbf{E}\eta_j^{(r)}) e^{itZ_n}],$$

and (2.1) of [18] holds, we get that the derivative with respect to t of the characteristic function $f_n(t)$ is equal to

$$\begin{aligned} f_n'(t) = i\Big[\sum_{j=1}^{n} a_j^{(1)} \mathbf{E}(\eta_j^{(2)} + 1) + \sum_{r=3}^{k} \sum_{j=1}^{n} a_j^{(r-1)} \mathbf{E}(\eta_j^{(r)} + 1)\Big] f_n(t) + i \sum_{j=1}^{n} \mathbf{E}(A_j e^{itz_j^{(1)}}) + \\ + i \sum_{j=2}^{k} \sum_{j=1}^{n} \Big[\mathbf{E}\Big(A_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{itz_j^{(r)}}\Big) - \mathbf{E}\Big(A_j \prod_{i=1}^{r-1} \xi_j^{(l)}\Big) \mathbf{E}\, e^{itz_j^{(r)}}\Big] + \\ + i \sum_{r=2}^{k} \sum_{j=1}^{n} a_j^{(r-1)} \mathbf{E}[(\eta_j^{(r)} - \mathbf{E}\,\eta_j^{(r)}) e^{itZ_n}] + i \sum_{j=1}^{n} \mathbf{E}\Big(A_j \prod_{l=1}^{k} \xi_j^{(l)} e^{itz_j^{(k)}}\Big). \end{aligned} \tag{3}$$

Tikhomirov (1980), Sunklodas (1984)

bound $\left|\phi(t) - \mathrm{e}^{-\sigma^2 t^2/2 + \mathrm{i}\mu}\right|$

set up differential equation for ϕ and bound its derivative

$$\langle \hat{X}_j \mathrm{e}^{\mathrm{i}\hat{\mathcal{X}}t} \rangle = \Big(\mathrm{i} \langle \hat{X}_j \mathcal{X} \rangle t + g(t) \Big) \varphi(t) + h(t),$$

where $g(t) = g_1(t) + g_2(t) + g_3(t)$, $h(t) = h_1(t) + h_2(t) + h_3(t)$,

$$g_1(t) = -\mathrm{i}\big(\langle \hat{X}_j \hat{z}_1 \rangle - \langle \hat{X}_j \rangle \langle \hat{z}_1 \rangle \big) t,$$

$$g_2(t) = \big\langle \hat{X}_j \hat{\xi}_1(t) \big\rangle + \mathrm{i} \langle \hat{X}_j \hat{z}_1 \rangle t - \mathrm{i} \langle \hat{X}_j \mathcal{X} \rangle t,$$

$$g_3(t) = \big\langle \hat{X}_j \hat{\xi}_1(t) \big\rangle \langle \hat{\eta}_2(t) \rangle + \sum_{n=3}^{k} \big\langle \hat{X}_j \hat{\Xi}_{n-1}(t) \big\rangle \langle (\hat{\eta}_n(t) + \mathbb{1}) \rangle,$$

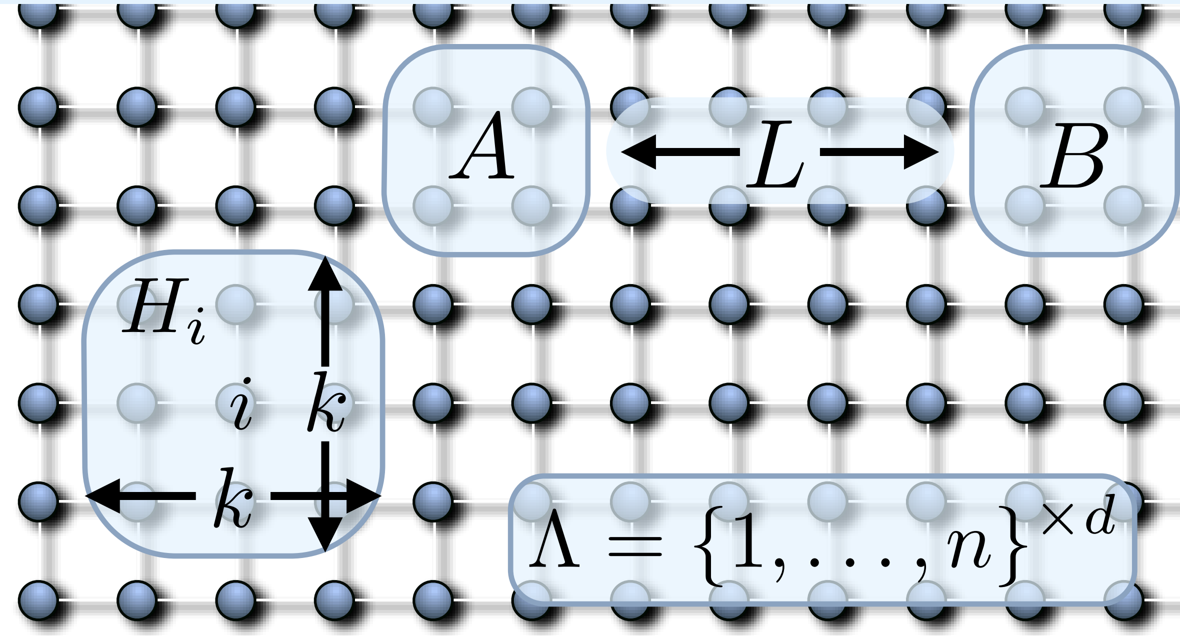
$$h_1(t) = \sum_{n=1}^{k} \left(\langle \hat{X}_j \hat{\Xi}_{n-1}(t) \mathrm{e}^{\mathrm{i}\hat{z}_n t} \rangle - \langle \hat{X}_j \hat{\Xi}_{n-1}(t) \rangle \langle \mathrm{e}^{\mathrm{i}\hat{z}_n t} \rangle \right),$$

$$h_2(t) = \sum_{n=2}^{k} \big\langle \hat{X}_j \hat{\Xi}_{n-1}(t) \big\rangle \big\langle \big(\hat{\eta}_n(t) - \langle \hat{\eta}_n(t) \rangle \big) \mathrm{e}^{\mathrm{i}\hat{\mathcal{X}}t} \big\rangle,$$

$$h_3(t) = \langle \hat{X}_j \hat{\Xi}_k(t) \mathrm{e}^{\mathrm{i}\hat{z}_k t} \rangle + \sum_{n=0}^{k-1} \langle \hat{X}_j \hat{\Xi}_n(t) \hat{r}_n(t) \mathrm{e}^{\mathrm{i}\hat{z}_{n+1} t} \rangle + \sum_{n=2}^{k} \big\langle \hat{X}_j \hat{\Xi}_{n-1}(t) \big\rangle \langle \hat{s}_n(t) \rangle,$$

$$\hat{r}_n(t) = \mathrm{e}^{\mathrm{i}(\hat{z}_n - \hat{z}_{n+1})t} \Big(\mathrm{e}^{-\mathrm{i}(\hat{z}_n - \hat{z}_{n+1})t} \mathrm{e}^{\mathrm{i}\hat{z}_n t} \mathrm{e}^{-\mathrm{i}\hat{z}_{n+1} t} - \hat{R}_{n+1}(t) \Big) =: \mathrm{e}^{\mathrm{i}(\hat{z}_n - \hat{z}_{n+1})} \Big(\hat{Z}_{R,n+1}(t) - \hat{R}_{n+1}(t) \Big)$$

$$\hat{s}_n(t) = \Big(\mathrm{e}^{-\mathrm{i}(-\hat{\mathcal{X}} + \hat{Z}_n)t} \mathrm{e}^{-\mathrm{i}\hat{\mathcal{X}}t} \mathrm{e}^{\mathrm{i}\hat{Z}_n t} - \hat{S}_n(t) \Big) \mathrm{e}^{-\mathrm{i}\hat{Z}_n t} \mathrm{e}^{\mathrm{i}\hat{\mathcal{X}}t} =: \Big(\hat{Z}_{S,n}(t) - \hat{S}_n(t) \Big) \mathrm{e}^{-\mathrm{i}\hat{Z}_n t} \mathrm{e}^{\mathrm{i}\hat{\mathcal{X}}t}.$$



$$H = \sum_{i\in\Lambda} H_i = \sum_k E_k |k\rangle\langle k|$$

canonical state $\varrho_T = \frac{\mathrm{e}^{-H/T}}{Z}$

with energy density $u(T) = \frac{\mathrm{tr}(H\varrho_T)}{N} \quad (= \frac{\mu}{N})$

specific heat capacity $c(T) = \frac{\partial u(T)}{\partial T} \quad (= \frac{\sigma^2}{NT^2})$

finite correlation length $\frac{|\langle AB\rangle - \langle A\rangle\langle B\rangle|}{\|A\|\|B\|} \le N^z \mathrm{e}^{-L/\xi}$

$d = 1$: Araki (1969)

$d > 1, T > T_c$: Kliesch, Gogolin, Kastoryano, Riera, Eisert (2014)

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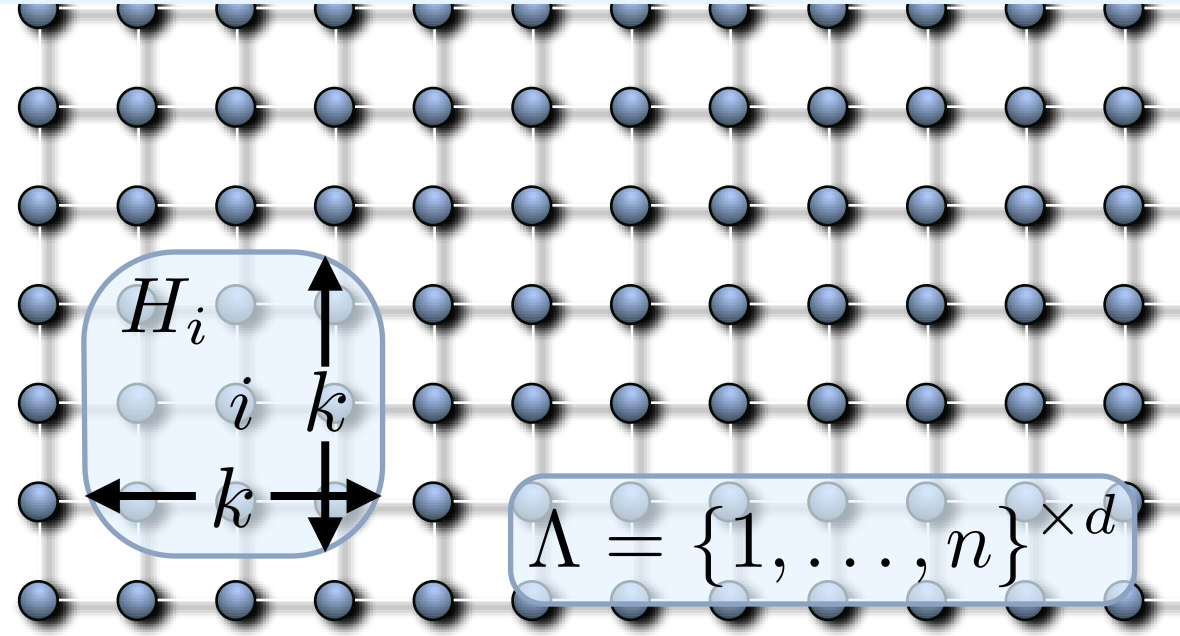


$$M_{e,\delta} = \left\{ k \,:\, |E_k - eN| \leq \delta\sqrt{N} \right\}$$

state ρ on subspace spanned by those $|k\rangle$

if $|e - u(T)| \leq \sqrt{\frac{c(T)T^2}{N}}$ and $\frac{C \ln^{2d}(N)}{\sqrt{N}} \leq \frac{\delta}{\sqrt{c(T)T^2}} \leq 1$

then

$$S(\rho \| \varrho_T) \leq \log(|M_{e,\delta}|) - S(\rho) + \left(\sqrt{c(T)}C + 4\right) \ln^{2d}(N)$$

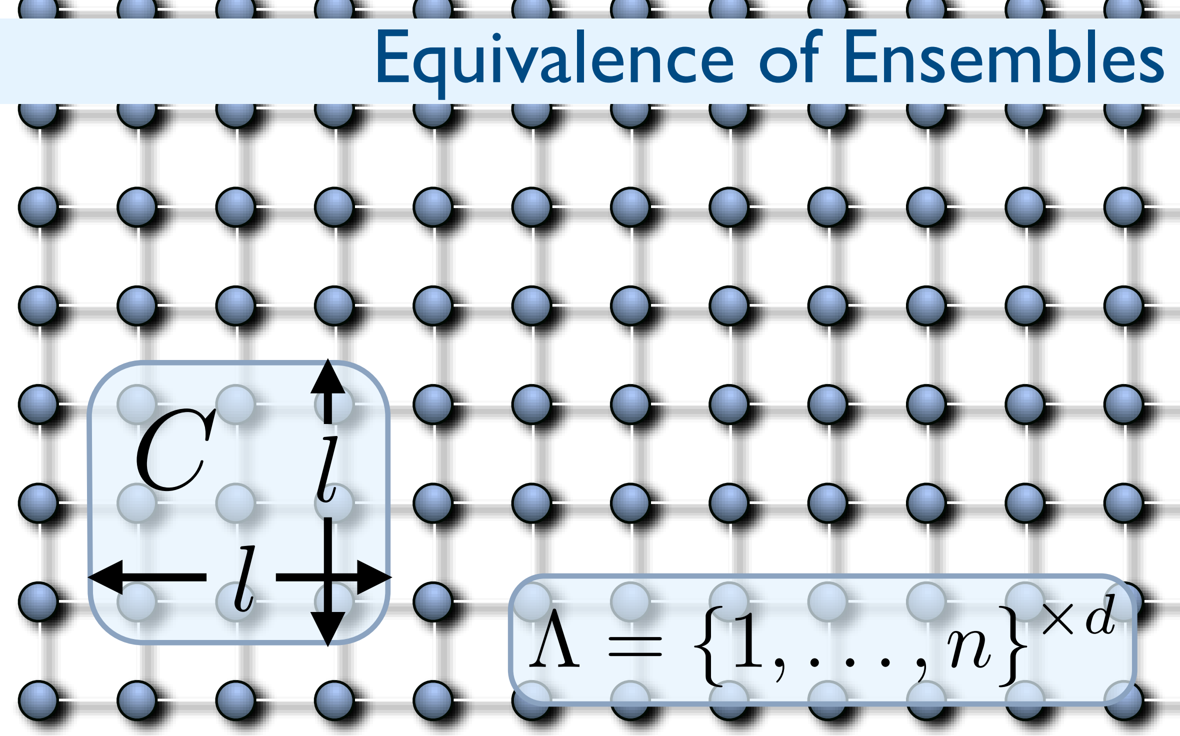
special case: microcanonical state $\frac{1}{|M_{e,\delta}|} \sum_{k \in M_{e,\delta}} |k\rangle\langle k|$

for which $S(\rho) = \log(|M_{e,\delta}|)$

$$\varrho = \varrho_T : \frac{|\langle AB\rangle - \langle A\rangle\langle B\rangle|}{\|A\|\|B\|} \leq N^z \mathrm{e}^{-L/\xi}$$

for which ρ (and which l) is

$\|\rho_C - \varrho_C\|_1$ small?



- Question goes back to Boltzmann and Gibbs
- Thermodynamic limit
  - Thermodynamical functions: Lebowitz, Lieb (1969); Lima (1971/72); Touchette (2009)
  - States: Mueller, Adlam, Masanes, Wiebe (2013)
- see also: Popescu, Short, Winter (2005); Riera, Gogolin, Eisert (2011)

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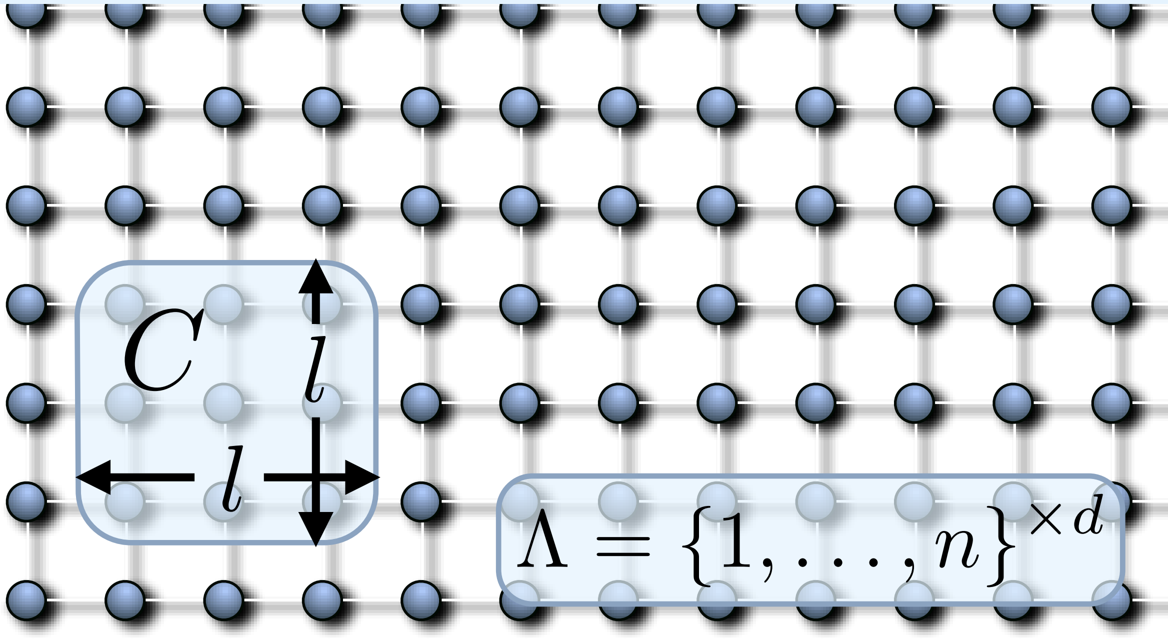


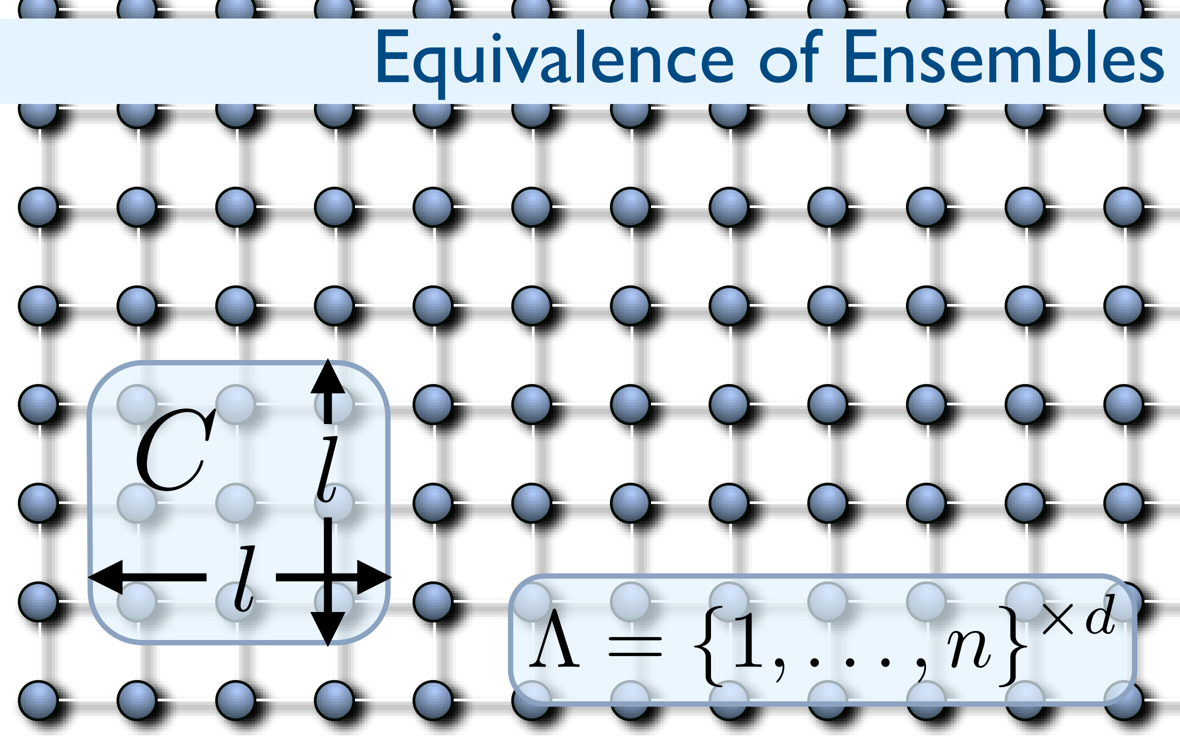
Here:

- Finite size, explicit bounds in system size
- More general states than microcanonical
- Equivalence of microcanonical states
- Not necessarily translational invariant

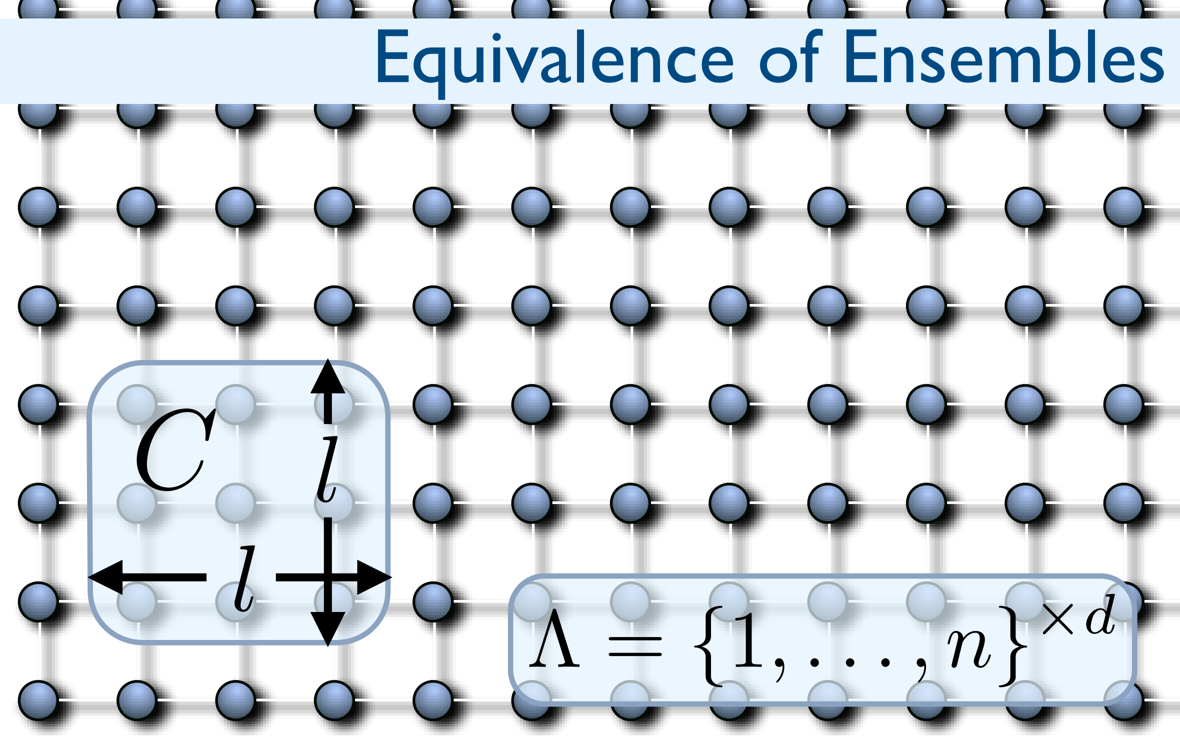
$$\varrho = \varrho_T : \frac{|\langle AB\rangle - \langle A\rangle\langle B\rangle|}{\|A\|\|B\|} \leq N^z \mathrm{e}^{-L/\xi}$$

for which ρ (and which l) is

$$\|\rho_C - \varrho_C\|_1 \leq 7\sqrt{\epsilon}$$?



non-t.i.: same holds for the expectation over all cubic regions of edge length l → by Markov's inequality $\mathbb{P}[\|\rho_C - \varrho_C\|_1 \geq a] \leq \frac{7\sqrt{\epsilon}}{a}$



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For microcanonical states

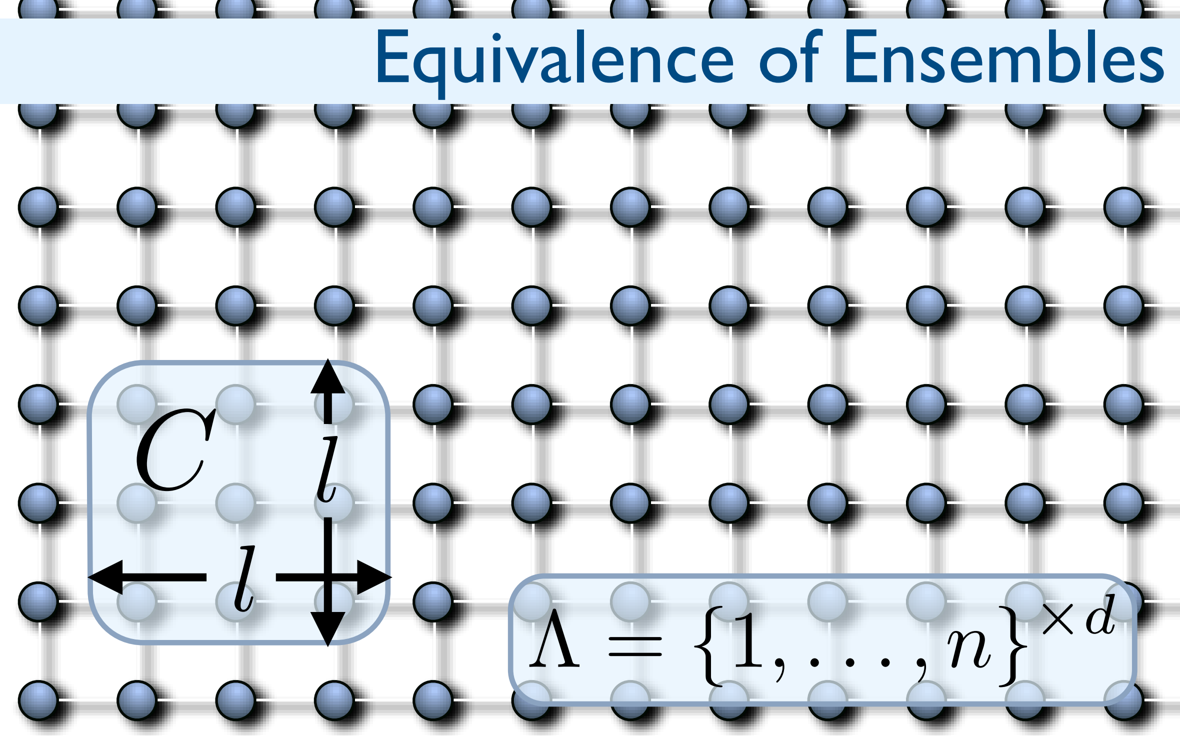
$$\rho = \frac{1}{|M_{e,\delta}|} \sum_{k \in M_{e,\delta}} |k\rangle\langle k| \text{ where } M_{e,\delta} = \left\{k \,:\, |E_k - eN| \leq \delta\sqrt{N}\right\}$$

with $|e - u(T)| \leq \sqrt{\frac{c(T)T^2}{N}}$ and $\frac{C \ln^{2d}(N)}{\sqrt{N}} \leq \frac{\delta}{\sqrt{c(T)T^2}} \leq 1$

and l such that $\frac{z+5+\sqrt{c(T)}C}{\epsilon\ln(2)} \ln^{2d}(N) + \frac{l+1+\xi d}{\xi} + l^d \leq \left(\frac{\epsilon N}{4^d \xi^d}\right)^{\frac{1}{d+1}}$

$\delta = 0$: Eigenstate Thermalization

$$\varrho = \varrho_T : \frac{|\langle AB\rangle - \langle A\rangle\langle B\rangle|}{\|A\|\|B\|} \leq N^z \mathrm{e}^{-L/\xi}$$



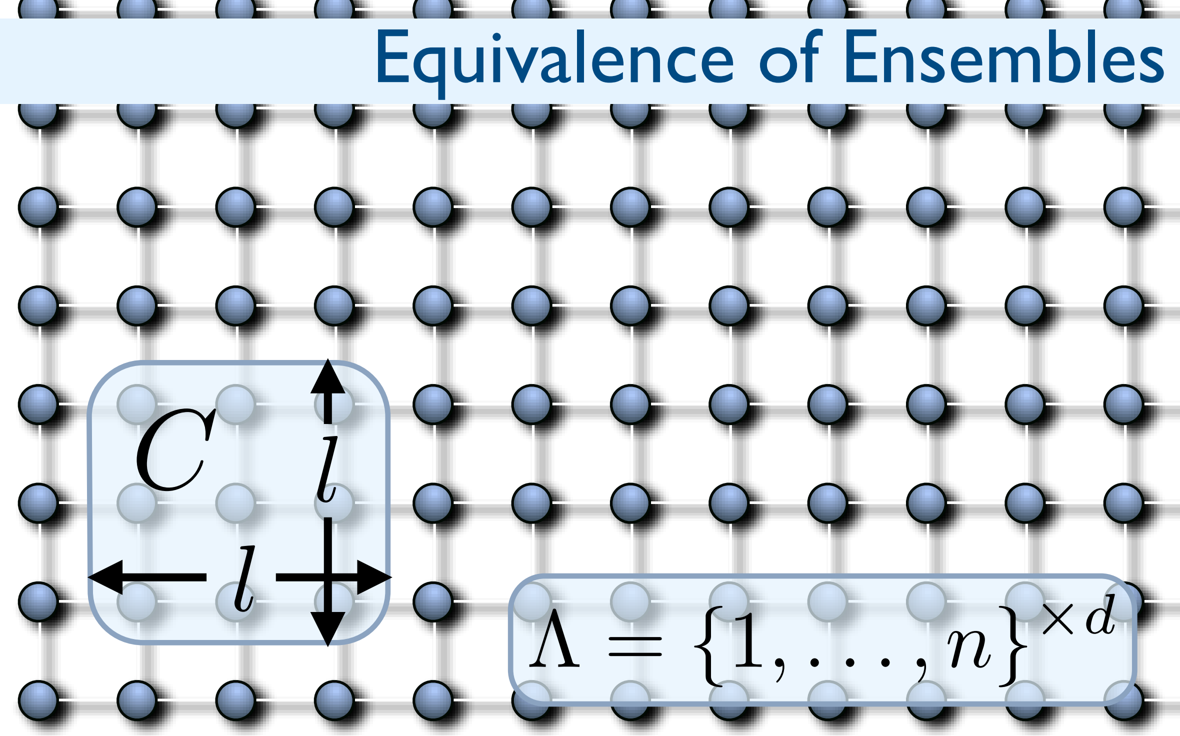
for which ρ (and which l) is

$\|\rho_C - \varrho_C\|_1$ small?

For pure states ρ drawn from subspace $\mathrm{span}\{|k\rangle\}_{k \in M_{e,\delta}}$:

$$\mathbb{P}\left[\|\rho_C - (\mathrm{m.c.})_C\|_1 \leq \sqrt{\epsilon} + \frac{2^{l^d}}{\sqrt{|M_{e,\delta}|}}\right] \geq 1 - 2\mathrm{e}^{-\frac{|M_{e,\delta}|\epsilon}{18\pi^3}}$$

Popescu, Short, Winter (2005)



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Popescu, Short, Winter (2005)

$$\underset{\text{QBE}}{\geq} 1 - 2\exp\left[-\frac{\epsilon}{18\pi^3}\exp\left(N\left(s(\varrho) - \frac{C\sqrt{c(T)}\ln^{2d}(N)}{\sqrt{N}}\right)\right)\right] =: p$$

$M_{e,\delta},\ e,\ \delta,\ l$ as before $\longrightarrow$ with prob. at least p:

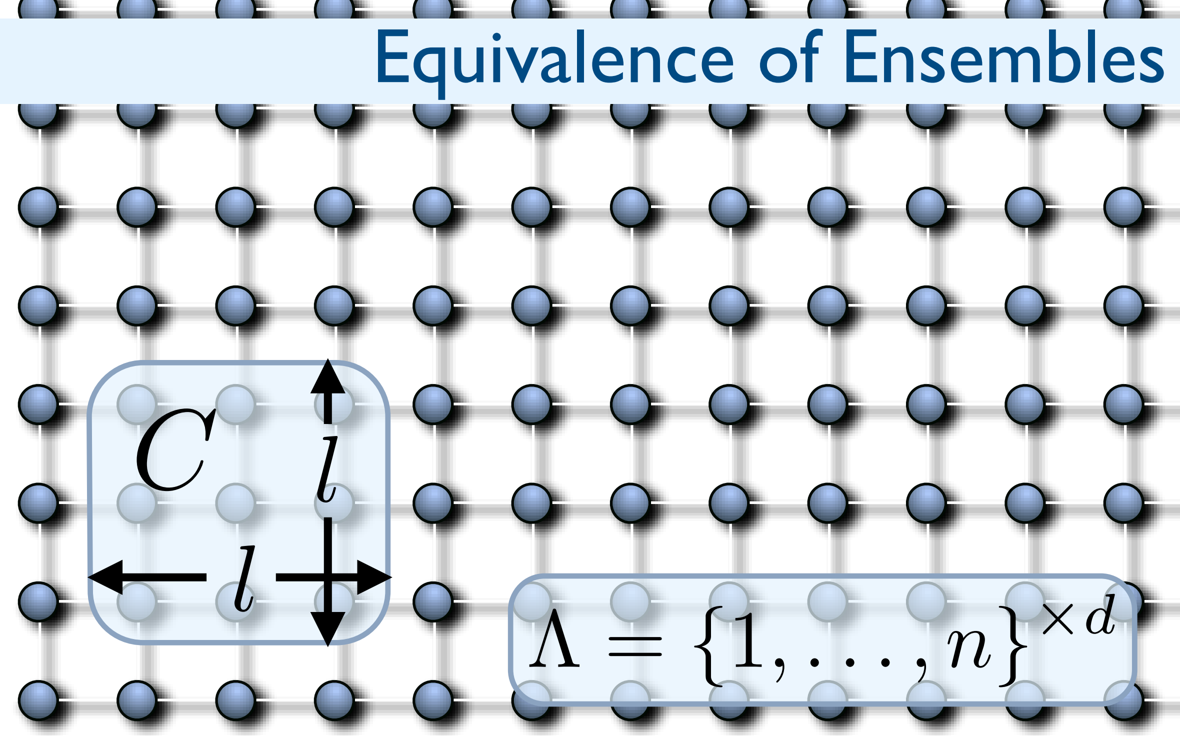
$$\|\rho_C - \varrho_C\|_1 \leq 8\sqrt{\epsilon} + 2^{l^d}\exp\left(-N\left(s(\varrho) - \frac{C\sqrt{c(T)}\ln^{2d}(N)}{\sqrt{N}}\right)/2\right)$$

cp. Riera, Gogolin, Eisert (2011); Mueller, Adlam, Masanes, Wiebe (2013)

$\varrho = \varrho_T : \frac{|\langle AB\rangle - \langle A\rangle\langle B\rangle|}{\|A\|\|B\|} \leq N^z \mathrm{e}^{-L/\xi}$

for which ρ (and which l) is

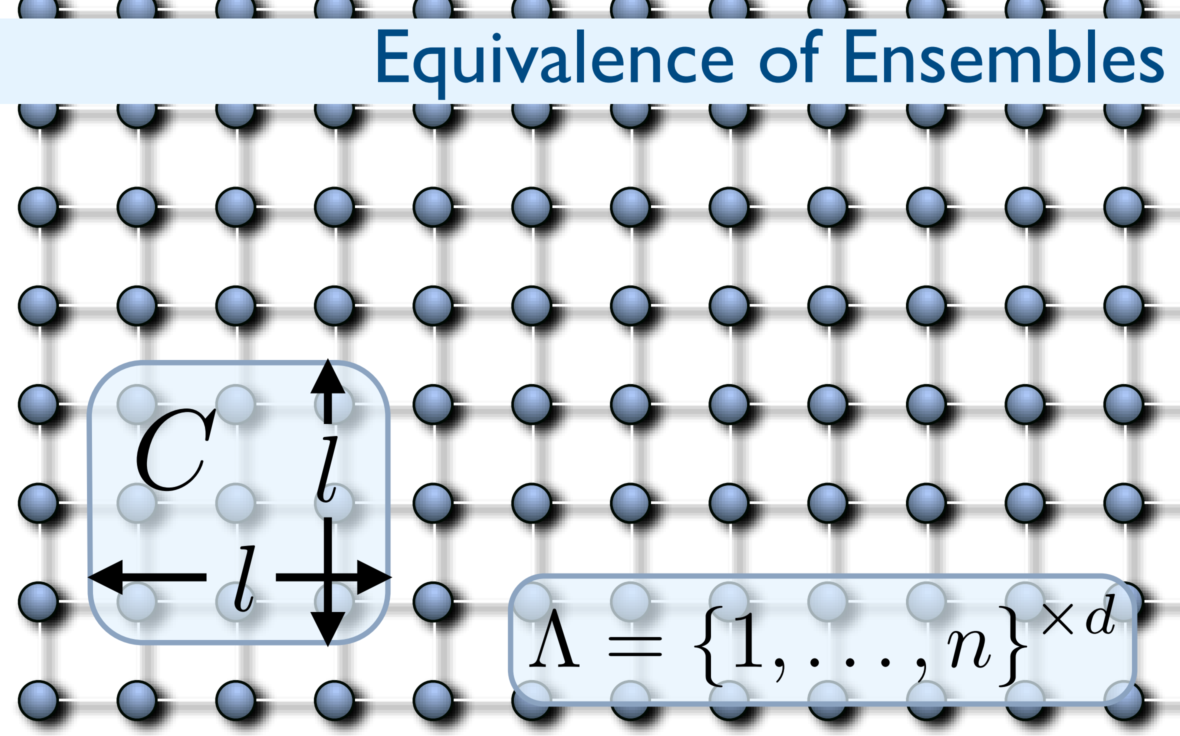
$$\|\rho_C - \varrho_C\|_1 \leq 7\sqrt{\epsilon} \quad ?$$



For those which fulfil

$$\frac{S(\rho\|\varrho_T) + 3}{\epsilon} + \frac{l + 1 + \xi d}{\xi} + l^d + \ln(N^{z+1}) \leq \left(\frac{\epsilon N}{4^d \xi^d}\right)^{\frac{1}{d+1}}$$

- quantum substate theorem Jain, Radhakrishnan, Sen (2009); Jain, Nayak (2011)
- Lemma Datta, Renner (2009); Brandão, Plenio (2010); Brandão, Horodecki (2012)
- Pinsker's inequality $\|\rho - \varrho\|_1^2 \leq \ln(4) S(\rho\|\varrho)$
- Super-additivity $\sum_{j=1}^{M} S(\rho_{C_j}\|\varrho_{C_j}) \leq S(\rho_{C_1 \cdots C_M}\|\varrho_{C_1 \cdots C_M})$
- $S(\rho\|\varrho) \leq S_{\max}(\rho\|\varrho)$ Datta (2009)



$\varrho = \varrho_T : \frac{|\langle AB\rangle - \langle A\rangle\langle B\rangle|}{\|A\|\|B\|} \leq N^z \mathrm{e}^{-L/\xi}$

for which ρ (and which l) is

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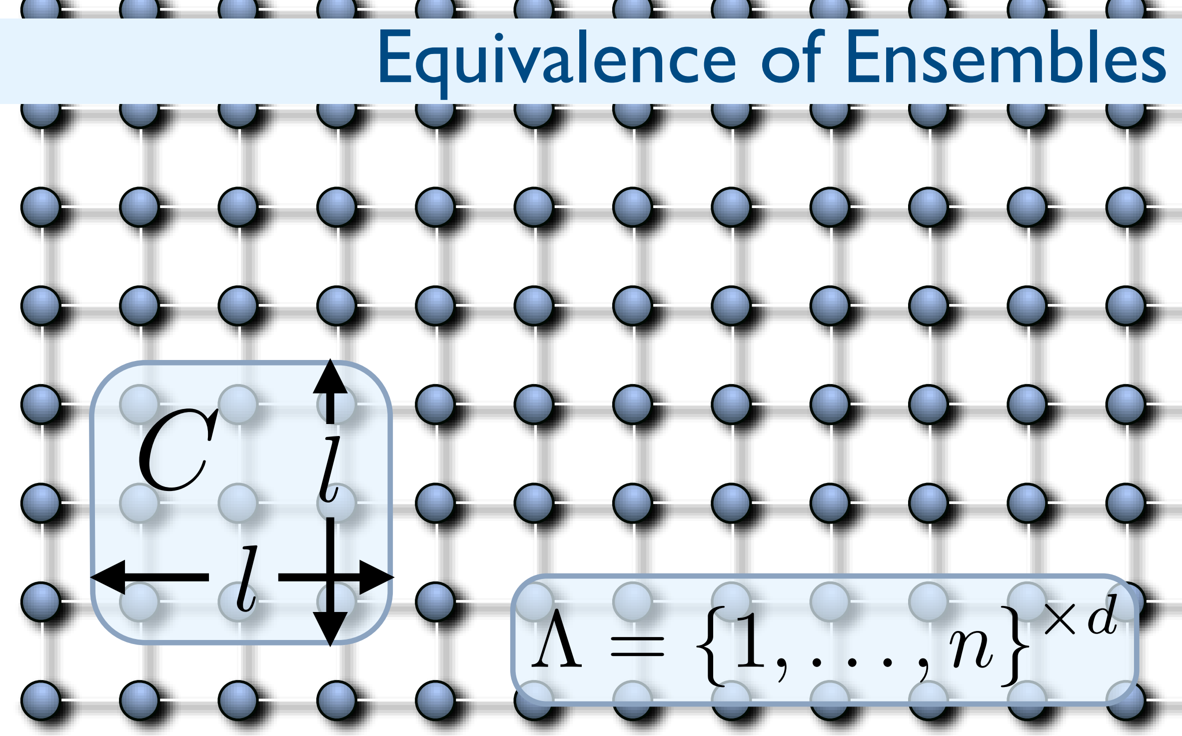
$$\frac{S(\rho\|\varrho_T)+3}{\epsilon} + \frac{l+1+\xi d}{\xi} + l^d + \ln(N^{z+1}) \leq \left(\frac{\epsilon N}{4^d \xi^d}\right)^{\frac{1}{d+1}}$$

i.e., as $TS(\rho\|\varrho_T) = F_T(\rho) - F_T(\varrho_T)$, it holds for states ρ with small free energy $F_T(\rho) = \mathrm{tr}[H\rho] - TS(\rho)$

cp. Th. 2 of Mueller, Adlam, Masanes, Wiebe (2013)

$\varrho = \varrho_T : \frac{|\langle AB \rangle - \langle A \rangle \langle B \rangle|}{\|A\| \|B\|} \leq N^z \mathrm{e}^{-L/\xi}$



States ρ are locally thermal ($\|\rho_C - \varrho_C\|_1$ is small) if

l as before and

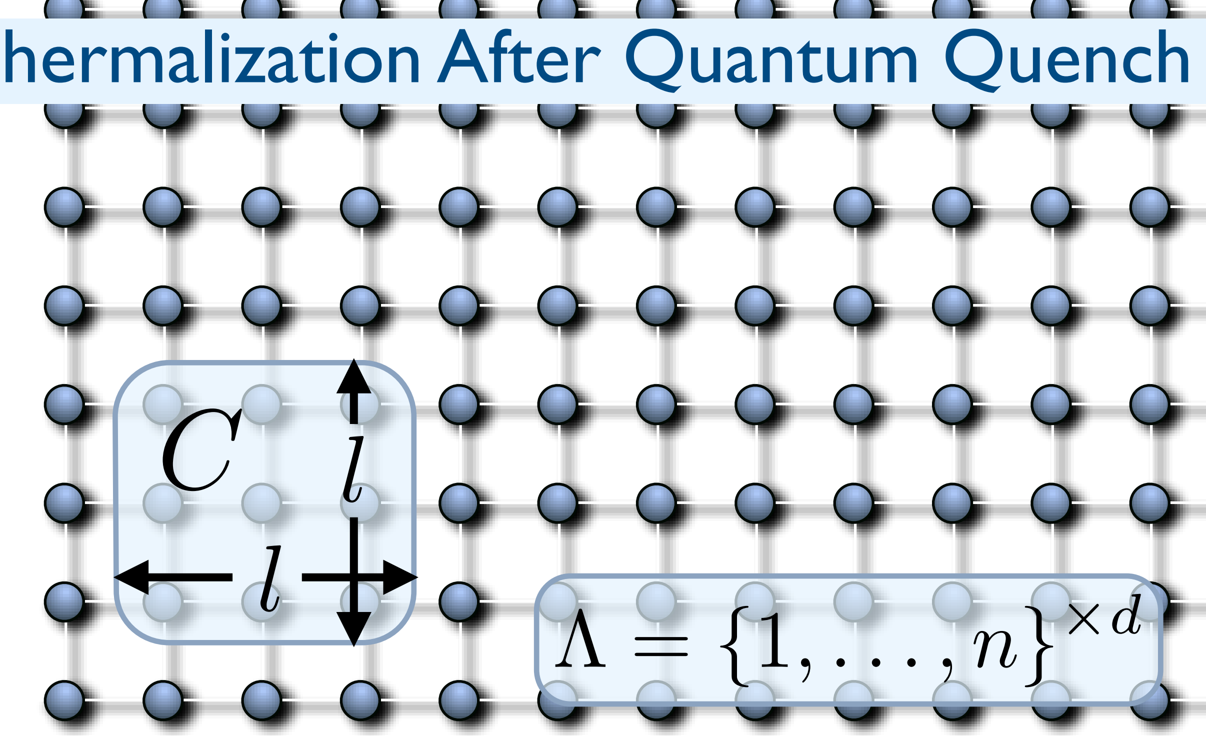
- $F_T(\rho) \leq F_T(\varrho) + T\epsilon \left(\frac{\epsilon N}{4^d \xi^d} \right)^{\frac{1}{d+1}}$

or

- ρ on the subspace corresponding to $M_{e,\delta}$ (as before) and

$$S(\rho) \geq \log(|M_{e,\delta}|) - \epsilon \left(\frac{\epsilon N}{4^d \xi^d} \right)^{\frac{1}{d+1}}$$

(in fact, “almost all” states in this subspace)



$$\rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\psi_0\rangle$$

$$H = \sum_k E_k |k\rangle\langle k|$$

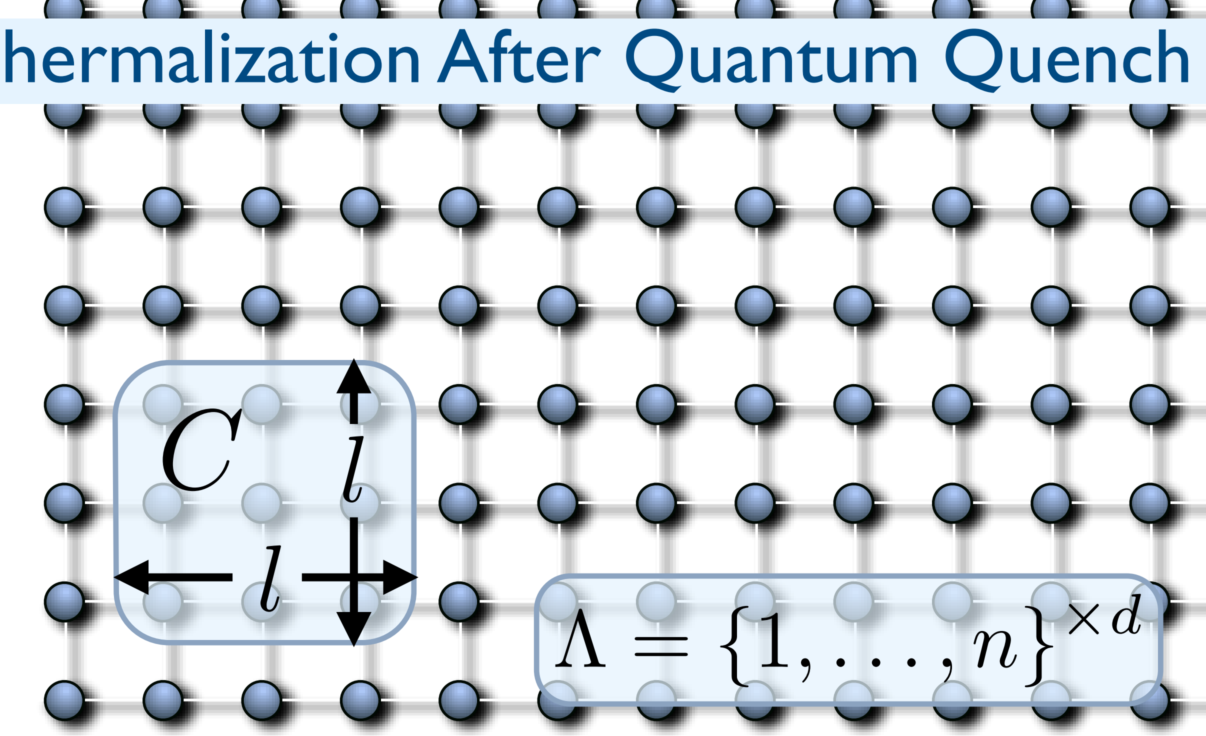
non-degen. energy gaps

$$\omega = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\, \rho(t) = \sum_k |\langle\psi_0|k\rangle|^2 |k\rangle\langle k|$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\, \|\rho(t) - \omega_C\|_1 \leq 2^{l^d}\sqrt{\mathrm{tr}[\omega^2]}$$

Linden, Popescu, Short, Winter (2008); Reimann (2008)

Local Thermalization After Quantum Quench



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non-degen. energy gaps, bounded and k-local

$$\omega = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\,\rho(t) = \sum_k |\langle\psi_0|k\rangle|^2 |k\rangle\langle k|$$

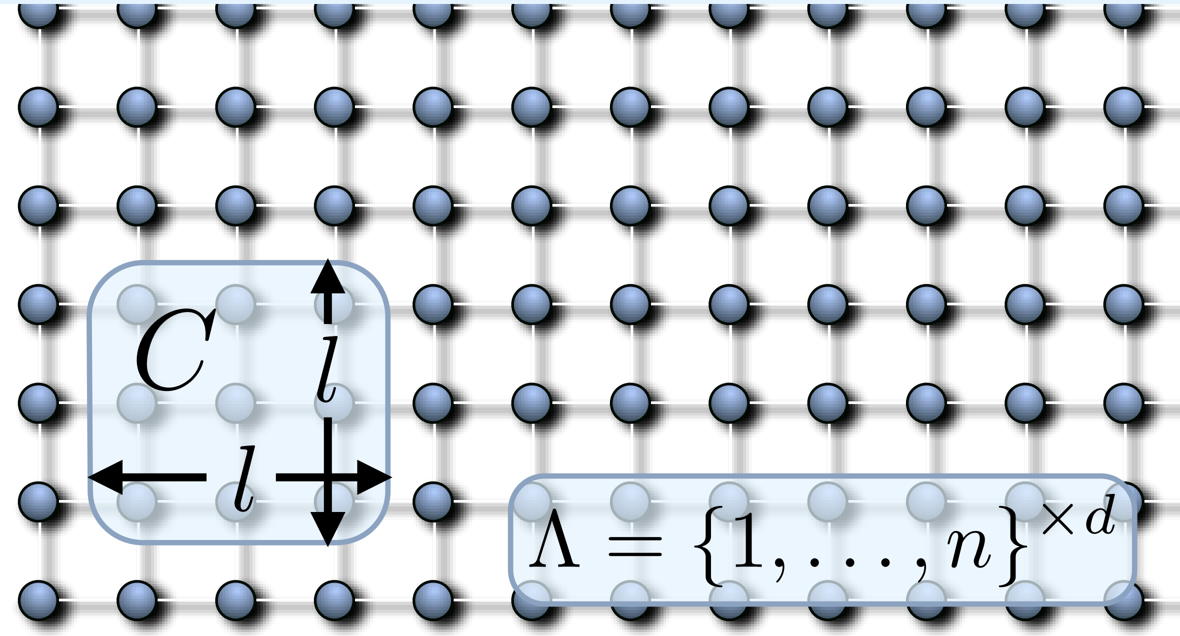
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\,\|\rho(t) - \omega_C\|_1 \leq 2^{l^d}\sqrt{\mathrm{tr}[\omega^2]}$$

$$\underset{|\psi_0\rangle \text{ as in QBE}}{\leq} 2^{l^d} C^{1/2}_{|\psi_0\rangle} \frac{\ln^d(N)}{N^{1/4}}$$

Linden, Popescu, Short, Winter (2008); Reimann (2008)

Local Thermalization After Quantum Quench

$$\varrho = \varrho_T : \frac{|\langle AB\rangle - \langle A\rangle\langle B\rangle|}{\|A\|\|B\|} \leq N^z \mathrm{e}^{-L/\xi}$$



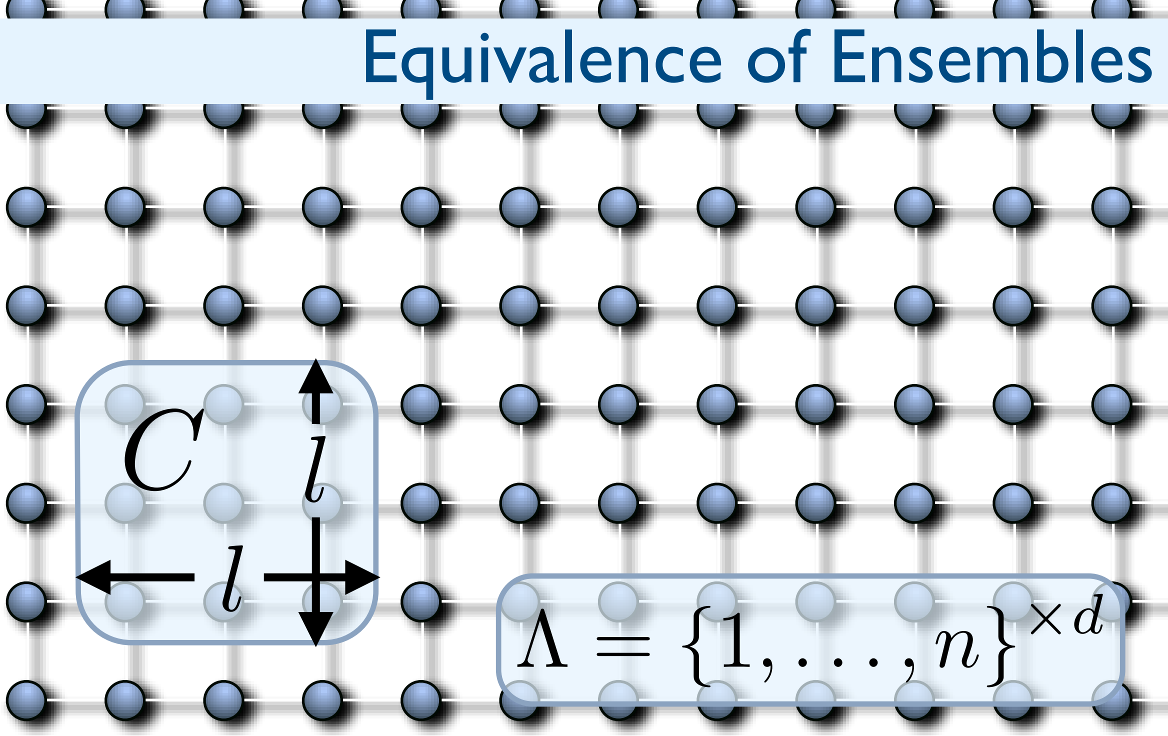
$$H = \sum_k E_k |k\rangle\langle k|$$

non-degen. energy gaps, bounded and k-local

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\, \|\rho(t) - \varrho_C\|_1 \leq 2^{l^d}\sqrt{2C_{|\psi_0\rangle}\frac{\ln^{2d}(N)}{\sqrt{N}}} + 7\sqrt{\epsilon}$$

Linden, Popescu, Short, Winter (2008); Reimann (2008)

$$\varrho = \varrho_T : \frac{|\langle AB\rangle - \langle A\rangle\langle B\rangle|}{\|A\|\|B\|} \leq N^z \mathrm{e}^{-L/\xi}$$



States ρ are locally thermal ($\|\rho_C - \varrho_C\|_1$ is small) if

l as before and

- $F_T(\rho) \leq F_T(\varrho) + T\epsilon \left(\frac{\epsilon N}{4^d \xi^d}\right)^{\frac{1}{d+1}}$

or

- ρ on the subspace corresponding to $M_{e,\delta}$ (as before) and

$$S(\rho) \geq \log(|M_{e,\delta}|) - \epsilon \left(\frac{\epsilon N}{4^d \xi^d}\right)^{\frac{1}{d+1}}$$

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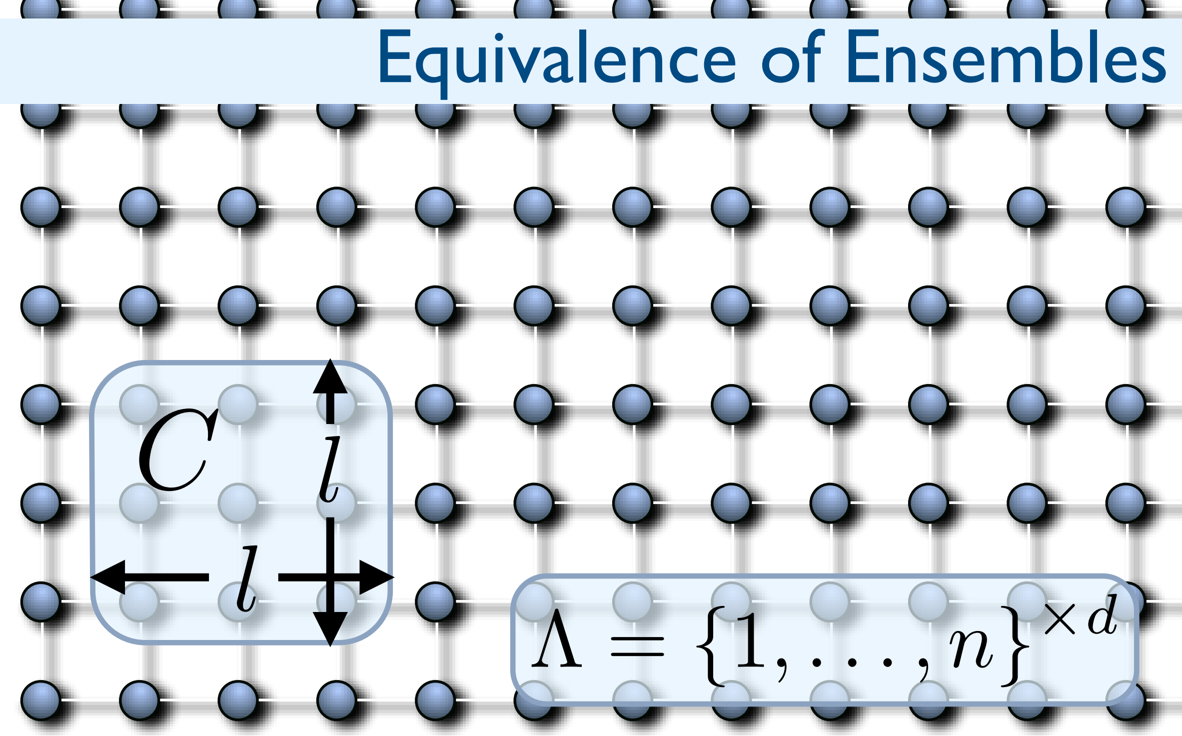

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(in fact, "almost all" states in this subspace)